

## Paranormal Elements in Normed Algebra

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Received March 29, 2017

**Abstract**—For a normed algebra  $\mathcal{A}$  and natural numbers  $k$  we introduce and investigate the  $\|\cdot\|$ -closed classes  $\mathcal{P}_k(\mathcal{A})$ . We show that  $\mathcal{P}_1(\mathcal{A})$  is a subset of  $\mathcal{P}_k(\mathcal{A})$  for all  $k$ . If  $T$  in  $\mathcal{P}_1(\mathcal{A})$ , then  $T^n$  lies in  $\mathcal{P}_1(\mathcal{A})$  for all natural  $n$ . If  $\mathcal{A}$  is unital,  $U, V \in \mathcal{A}$  are such that  $\|U\| = \|V\| = 1$ ,  $VU = I$  and  $T$  lies in  $\mathcal{P}_k(\mathcal{A})$ , then  $UTV$  lies in  $\mathcal{P}_k(\mathcal{A})$  for all natural  $k$ . Let  $\mathcal{A}$  be unital, then 1) if an element  $T$  in  $\mathcal{P}_1(\mathcal{A})$  is right invertible, then any right inverse element  $T^{-1}$  lies in  $\mathcal{P}_1(\mathcal{A})$ ; 2) for  $\|I\| = 1$  the class  $\mathcal{P}_1(\mathcal{A})$  consists of normaloid elements; 3) if the spectrum of an element  $T$ ,  $T \in \mathcal{P}_1(\mathcal{A})$  lies on the unit circle, then  $\|TX\| = \|X\|$  for all  $X \in \mathcal{A}$ . If  $\mathcal{A} = \mathcal{B}(\mathcal{H})$ , then the class  $\mathcal{P}_1(\mathcal{A})$  coincides with the set of all paranormal operators on a Hilbert space  $\mathcal{H}$ .

**DOI:** 10.3103/S1066369X1805002X

**Keywords:** Hilbert space,  $C^*$ -algebra, paranormal operator, quasinilpotent operator, isometry, hyponormal operator, normaloid operator, normed algebra, unital algebra.

**Introduction.** Investigation of different subsets in normed algebras and in  $*$ -algebras of operators is an actual problem of functional analysis (see, e.g., [1–5] for classes of hyponormal, normal, idempotent, unitary operators and differences of idempotents, respectively). In this paper for a normed algebra  $\mathcal{A}$  and  $k \in \mathbb{N}$  we introduce and investigate  $\|\cdot\|$ -closed classes

$$\mathcal{P}_k(\mathcal{A}) = \{T \in \mathcal{A} : \|T^{k+1}A\| \geq \|TA\|^{k+1} \text{ for all } A \in \mathcal{A} \text{ with } \|A\| = 1\}.$$

It is shown that  $\mathcal{P}_1(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{A})$  for all  $k \in \mathbb{N}$  (Theorem 2). If  $\mathcal{A}$  is a dense subalgebra of normed algebra  $\mathcal{B}$ , then  $\mathcal{P}_k(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{B})$  for all  $k \in \mathbb{N}$  (Proposition 1). If  $T \in \mathcal{P}_1(\mathcal{A})$ , then  $T^n \in \mathcal{P}_1(\mathcal{A})$  for all  $n \in \mathbb{N}$  (Theorem 5). If  $\mathcal{A}$  is unital,  $U, V \in \mathcal{A}$  are such that  $\|U\| = \|V\| = 1$ ,  $VU = I$  and  $T \in \mathcal{P}_k(\mathcal{A})$ , then  $UTV \in \mathcal{P}_k(\mathcal{A})$  for all  $k \in \mathbb{N}$  (Theorem 3). In particular, if  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $T \in \mathcal{P}_k(\mathcal{A})$ , then  $UTU^* \in \mathcal{P}_k(\mathcal{A})$  for all isometries  $U \in \mathcal{A}$  and  $k \in \mathbb{N}$  (Corollary 3). If  $\mathcal{A}$  is commutative and  $\|T^2\| = \|T\|^2$  for all  $T \in \mathcal{A}$ , then  $\mathcal{P}_1(\mathcal{A}) = \mathcal{A}$  (Proposition 6).

Let  $\mathcal{A}$  be unital, then 1) if an element  $T \in \mathcal{P}_1(\mathcal{A})$  is right invertible, then any right inverse element  $T^{-1}$  lies in  $\mathcal{P}_1(\mathcal{A})$  (Theorem 4); 2) for  $\|I\| = 1$  the class  $\mathcal{P}_1(\mathcal{A})$  consists of normaloid elements (Corollary 1); 3) if the spectrum of an element  $T$ ,  $T \in \mathcal{P}_1(\mathcal{A})$  lies on the unit circle, then  $\|TX\| = \|X\|$  for all  $X \in \mathcal{A}$  (Corollary 4). If  $\mathcal{A} = \mathcal{B}(\mathcal{H})$ , then the class  $\mathcal{P}_1(\mathcal{A})$  coincides with the set of all paranormal operators on a Hilbert space  $\mathcal{H}$  (Corollary 6).

**1. Notations and definitions.** An algebra is a vector space  $\mathcal{A}$  over the field  $\Lambda$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ), equipped with a bilinear product such that

$$X(YZ) = (XY)Z, \quad (Y + Z)X = YX + ZX,$$

$$X(Y + Z) = XY + XZ, \quad \lambda(XY) = (\lambda X)Y = X(\lambda Y)$$

for all  $X, Y, Z \in \mathcal{A}$  and  $\lambda \in \Lambda$ . An algebra  $\mathcal{A}$  is *unital* (i.e., possesses the unity), if there exists an element  $(0 \neq)I \in \mathcal{A}$  such that  $IX = XI = X$  ( $X \in \mathcal{A}$ ). An element  $X$  of algebra  $\mathcal{A}$  with  $I$  is said to be *right invertible*, if there exists an element  $X^{-1} \in \mathcal{A}$  such that  $XX^{-1} = I$ . An algebra  $\mathcal{A}$  is said to

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